B.A./B.Sc. 4th Semester (General) Examination, 2019

Subject : Mathematics

Paper : BMG4CC1D and Math-GE4

(Algebra)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

[Symbols and notation have their usual meaning.]

1. Answer *any ten* questions: 2×10=20

(a) Does the set of rational numbers form a group with respect to usual multiplication? Justify your answer.

(b) Consider the group $(G, *)$. If for all $a, b, c \in G$, $a * c = b * c$, prove that $a = b$.

(c) Show that a group G is abelian if $(ab)^2 = a^2b^2\ \forall a, b \in G$.

(d) If $(G, *)$ be a group such that $a = a^{-1}\ \forall a \in G$, then show that $(G, *)$ is an abelian group.

(e) Define cyclic group and give an example of a cyclic group.

(f) Show that the set E of all even integers is a subgroup of the additive group $(\mathbb{Z}, +)$ of integers.

(g) Show that in any ring R, $a.0 = 0\ \forall a \in R$.

(h) Define integral domain. Give an example of a finite integral domain.

(i) Show the ring of integers $(Z, +, \cdot)$ is not a field.

(j) Prove that any field is an integral domain.

(k) Prove that the group $(\mathbb{Q}, +)$ is not cyclic.

(l) Prove that union of two subgroups of a group G may not be subgroup of G.

(m) Prove that every cyclic group is abeliam.

(n) Prove that A_3 is a normal subgroup of S_3 with usual meaning.

(o) $\mathbb{Z}$ is an ideal of $(9\mathbb{Z}, +, \cdot)$— Justify. (Symbols have usual meanings).

5×4=20

2. Answer *any four* questions:

(a) Show that the intersection of two subgroups of a group G is a subgroup of G. Is the union of two subgroups of a group G a subgroup of G? 3+2=5

(b) State and prove Lagrange's theorem for finite group. 1+4=5

(c) Let M be the ring of all 2×2 matrics over integers and $L = \left\{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}; a, b \in \mathbb{Z}\right\}$.

Show that L is a left ideal of M but not a right ideal of M. 3+2=5

(d) Show that ring $\mathbb{Z}_n$ of integers modulo n is a field iff n is a prime. 2+3=5

(e) (i) Prove that intersection of two normal subgroups of a group G is a normal subgroup of G.

(ii) Let G be a group such that every cyclic subgroup of G is a normal subgroup of G. Prove that every subgroup of G is a normal subgroup of G. 3+2=5

(f) Prove that $(\mathbb{Z}, +, \cdot)$ is a ring with usual meaning. 5

(g) (i) Define ideal of a ring.

(ii) Prove that the set I of all integers is a subring of the ring $\mathbb{Q}$ of rationals but I is not an ideal of $\mathbb{Q}$. 1+4=5

3. Answer *any two* questions: 2×10=20

(a) (i) Consider $GL(2,\mathbb{R}) = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R}, ad - bc \neq 0\right\}$.

Define $\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} ax + bz & ay + bt \\ cx + dz & cy + dt \end{bmatrix}$ for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} x & y \\ z & t \end{bmatrix} \in GL(2,\mathbb{R})$.

Prove that $(GL(2,\mathbb{R}), *)$ is a non-abelian group.

(ii) Consider $GL(2,\mathbb{R})$ as in (i) and $SL(2,\mathbb{R}) = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,\mathbb{R}) : ad - bc = 1\right\}$.

Prove that $SL(2,\mathbb{R})$ is a subgroup of $GL(2,\mathbb{R})$. Is it a normal subgroup of $GL(2,\mathbb{R})$? 5+5=10

(b) (i) Prove that the roots of the equation $x^6 - 1 = 0$ form a subgroup of the multiplicative group of non-zero complex numbers. Is the subgroup cyclic?

(ii) Prove that union of two subgroups of a group G is a subgroup of G if and only if any one of them is contained in other. (4+1)+(4+1)=10

(c) (i) Show that the set of all non-zero elements of a field forms a commutative group under multiplication.

(ii) Show that the set $R = \{a + b\omega : a, b, \in \mathbb{R}\}$ forms a field with respect to usual addition and multiplication of complex numbers, where ω is a cube root of unity and $\mathbb{R}$ is the set of all real numbers. 5+5=10

(d) (i) Let G be a group. Then prove that (I) $(a^{-1})^{-1} = a \; \forall a \in G$ and (II) $(ab)^{-1} = b^{-1}a^{-1}, \forall a, b \in G$.

(ii) Let G be the set of all matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ where a, b, c are real numbers such that $ac \neq 0$. Prove that G forms a subgroup of $GL_2(\mathbb{R})$. 5+5=10

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